Higher Invariants
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Inner-amenable groups and homology growth

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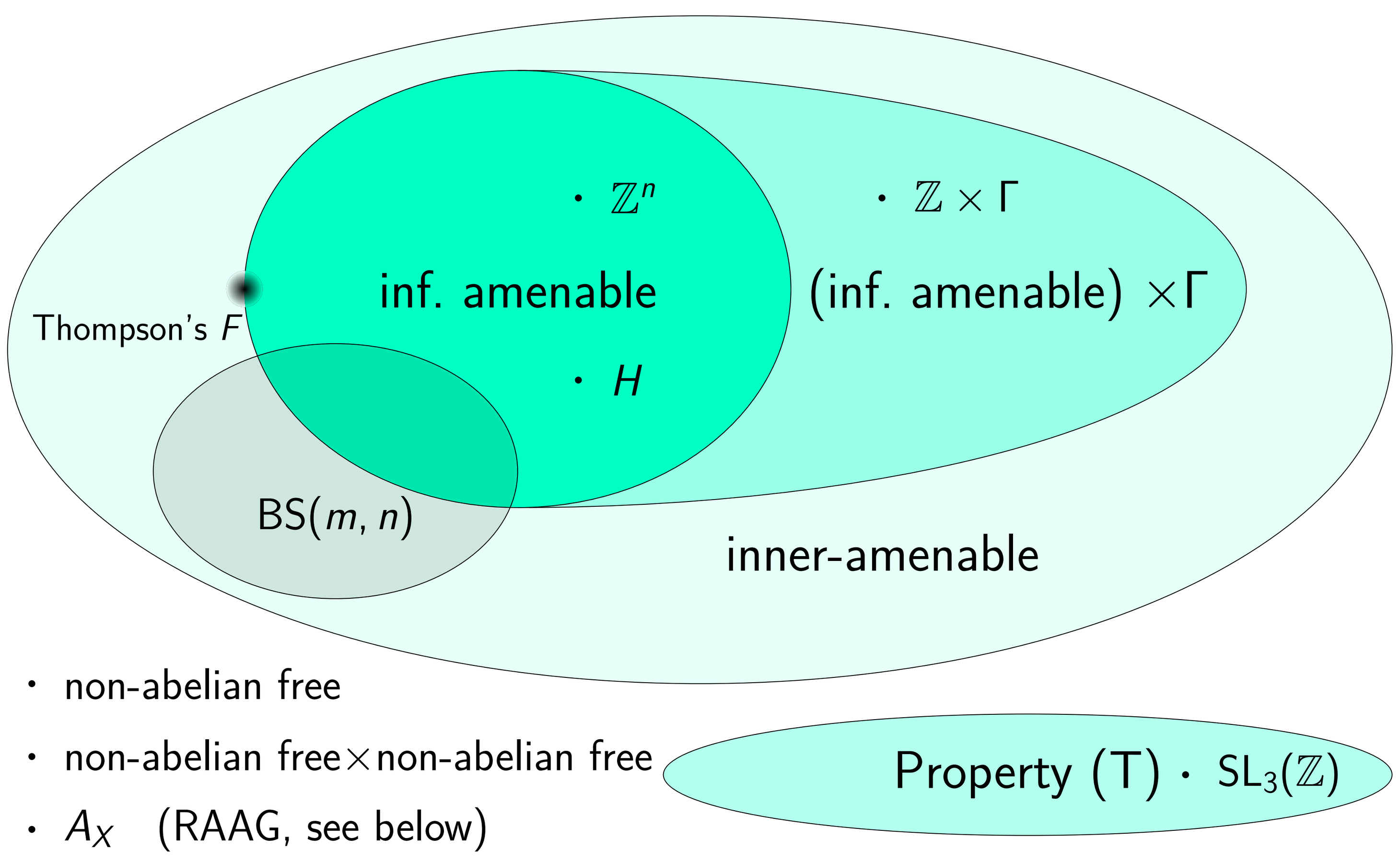
What is inner-amenability?

Definition. A group Γ is inner-amenable if there exists an atomless conjugation-invariant mean on Γ, i.e. a finitely additive probability measure $m : \mathcal{P}(\Gamma) \to [0,1]$ such that for all $\gamma \in \Gamma$, $A \subset \Gamma$, we have

$$m(\{\gamma\}) = 0 \quad \text{and} \quad m(\gamma \cdot A \cdot \gamma^{-1}) = m(A).$$

Examples.



Here, let X be a flag triangulation of $\mathbb{R}P^2$ [2]. The group H is a solvable, finitely generated, not finitely presented example [4].

What is homology growth?

Let Γ be residually finite and countable. Let $(\Gamma_i)_{i\in\mathbb{N}}$ be a residual chain, i.e. a sequence of finite-index, normal subgroups such that

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \dots \quad \text{and} \quad \bigcap_{i\in\mathbb{N}} \Gamma_i = \{1\}.$$

Definition. Let $n \in \mathbb{N}$ and K be a field. We define

$$\hat{b}_n(\Gamma, (\Gamma_i)_i, K) := \limsup_{i\to\infty} \frac{\dim_K H_n(B\Gamma_i, K)}{[\Gamma : \Gamma_i]}$$
$$\hat{t}_n(\Gamma, (\Gamma_i)_i) := \limsup_{i\to\infty} \frac{\log|\operatorname{tor} H_n(B\Gamma_i, \mathbb{Z})|}{[\Gamma : \Gamma_i]}.$$

Examples.

group	$\hat{b}_1(\cdot, \mathbb{Q})$	$\hat{b}_2(\cdot, \mathbb{Q})$	$\hat{b}_2(\cdot, \mathbb{F}_2)$	$\hat{t}_1$	$\hat{t}_2$
$\mathbb{Z}^n$	0	0	0	0	0
$\mathbb{Z} \times \Gamma$	0	0	0	0	0
F_{r+1}	r	0	0	0	0
$F_{r+1} \times F_{s+1}$	0	$r \cdot s$	$r \cdot s$	0	0
A_X	0	0	1	0	> 0
H	0	?	?	> 0	?

Theorem (U. [6]). Let Γ be a finitely presented, inner-amenable, torsion-free and residually finite group. Then, for every field K and every residual chain $(\Gamma_i)_i$, we have

$$\hat{b}_1(\Gamma, (\Gamma_i)_i, K) = 0 \quad \text{and} \quad \hat{t}_1(\Gamma, (\Gamma_i)_i) = 0.$$

Idea of proof. Use a structure theorem for inner-amenable groups [5] and inheritance of the cheap rebuilding property [1].

References

[1] M. Abert, N. Bergeron, M. Fraczyk and D. Gaboriau. On homology torsion growth. J. Eur. Math. Soc. (2024), published online first.
[2] G. Avramidi, B. Okun and K. Schreve. Mod p and torsion homology growth in nonpositive curvature. Invent. Math. 226, No. 3, 711–723 (2021).
[3] J. Cheeger and M. Gromov. L_2-cohomology and group cohomology. Topology 25, 189–215 (1986).
[4] A. Kar, P. Kropholler and N. Nikolov. On growth of homology torsion in amenable groups. Math. Proc. Camb. Philos. Soc. 162, No. 2, 337–351 (2017).
[5] R. D. Tucker-Drob. Invariant means and the structure of inner amenable groups. Duke Math. J. 169, No. 13, 2571–2628 (2020).
[6] M. Uschold. Torsion homology growth and cheap rebuilding of inner-amenable groups. To appear in Groups, Geometry and Dynamics.